

Topology of generic holomorphic foliations on Stein manifolds: structure of leaves and Kupka-Smale property.

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Abstract

We study topology of leaves of 1-dimensional singular holomorphic foliations of Stein manifolds. We prove that for a generic foliation all leaves, except for at most countably many are contractible, the rest are topological cylinders. We show that a generic foliation is complex Kupka-Smale.

1 Introduction

Consider a system of differential equations

$$\begin{aligned} x'_1 &= f_1(x_1, \dots, x_n) \\ &\vdots \\ x'_n &= f_n(x_1, \dots, x_n) \end{aligned} \quad (1)$$

where $(x_1, \dots, x_n) \in \mathbb{C}^n$, $f_1, \dots, f_n \in \mathcal{O}(\mathbb{C}^n)$.

The phase space \mathbb{C}^n , outside the singular locus, is foliated by Riemann surfaces. It is a natural question: what is the topological type of these leaves? For polynomial foliations of fixed degree this question was asked by Anosov and still remains unsolved. In general, it can be quite complicated. Consider, for example, a Hamiltonian foliation of \mathbb{C}^2 : $H_n = \text{const}$, where H_n is a generic polynomial of degree n . All non-singular leaves are Riemann surfaces with $\frac{(n-1)(n-2)}{2}$ handles and n punctures. There are examples of foliations with dense leaves, having infinitely generated fundamental groups [18].

So one can restrict the question: what is the topological type of leaves for a generic foliation?

The genericity here is understood as follows: the space of holomorphic foliations can be naturally equipped with the (Baire) topology of uniform convergence on nonsingular compacts sets. We recall the definition of the topology in Appendix 5.4. We call a foliation generic if it belongs to a residual set – an intersection of countably many open everywhere dense sets.

In our paper we describe the topological type of leaves for generic foliations on \mathbb{C}^n , and more generally, on arbitrary Stein manifolds. We prove the following theorem:

Theorem 1.1. *For a generic 1-dimensional singular holomorphic foliation on a Stein manifold X all leaves, except for at most countably many, are contractible, the rest are topological cylinders.*

We consider foliations with singular locus of codimension 2, i.e. foliations locally determined by holomorphic vector fields [15].

Our technique is applicable in a more general setting. In particular, we establish the analog of Kupka-Smale theorem for generic foliations on Stein manifolds:

Theorem 1.2. *A generic 1-dimensional singular holomorphic foliation on X is complex Kupka-Smale.*

Definition 1.1. A foliation of a complex manifold is called *complex Kupka-Smale* if

1. all its singular points are complex hyperbolic;
2. all complex cycles are hyperbolic;
3. strongly invariant manifolds of different singular points intersect transversally;
4. invariant manifolds of complex cycles intersect transversally with each other and with strongly invariant manifolds of singular points.

Let cycle γ be a phase curve of a real vector field, then γ is a loop on the phase curve of the complexified vector field. A complex cycle by definition is a free homotopy class of loops on a leaf of a foliation. Recall that by definition, a real Kupka-Smale vector field has hyperbolic cycles only. Condition (2) is a generalization of this property.

We review notions of complex hyperbolicity and invariant manifolds in the Appendix.

The above definition was suggested by Marc Chaperon in [4]. In this preprint he studies holomorphic 1-dimensional singular foliations on Stein manifolds. He shows that the property (1) holds for generic foliations. He also gives the proof of the property (3) for generic foliations on \mathbb{C}^n and states the result for generic foliations on Stein manifolds. Our technique also allows us to prove transversality results for strongly invariant manifolds of the same singular point:

Theorem 1.3. *For a generic 1-dimensional singular holomorphic foliation:*

1. *all singular points are complex hyperbolic.*
2. *Let a_1 be a complex hyperbolic singular point of the foliation. Let M_1 and M_2 be strongly invariant manifolds of the point a_1 , such that $M_1^{loc} \cap M_2^{loc} = a_1$. Then M_1 and M_2 intersect transversally everywhere.*

Theorems 1.1, 1.2 for foliations of \mathbb{C}^2 are proved in [7]. Golenishcheva-Kutuzova [10] showed that for a generic foliation countable many cylinders do exist. We expect that for a generic singular holomorphic 1-dimensional foliation of a Stein manifold there are countably many cylinders.

The conformal type of leaves of a generic polynomial foliation of fixed degree was described by Candel, Gomez-Mont [6]. The result was later improved by Lins Neto [17], and Glutsyuk[9]:

Theorem 1.4. [9], [17] *Any leaf of a generic polynomial foliation of degree n is hyperbolic.*

We expect that the same answer is true for generic foliations of Stein manifolds. The technique from [6], [17], [9] can be adjusted to attack the problem. See the paper [14] for a vast discussion of open problems.

Greg Buzzard studied similar genericity questions for polynomial automorphisms of \mathbb{C}^n . He proved that a generic polynomial automorphism of \mathbb{C}^n is Kupka-Smale [2].

1.1 Outline of the article.

We establish generic properties of foliations by constructing perturbations that eliminate degeneracies. There are at most countably many isolated cycles. (This lemma is proved in [16] for foliations of \mathbb{C}^2 . We included the proof for arbitrary Stein manifolds in Section 4.1 to explain

our strategy of simultaneous elimination of degeneracies.) Therefore, once all nonisolated cycles are removed, all leaves, except for countably many, are contractible.

To prove that the rest have fundamental group \mathbb{Z} , one needs to eliminate all degeneracies from the following list:

1. two cycles that belong to the same leaf of the foliation and are not multiples of the same cycle in the homology group of the leaf;
2. saddle connections;
3. cycles on a separatrix that are not multiples of the cycle around the critical point.

Recall that a separatrix is a leaf that can be holomorphically extended into a singular point and a saddle connection is a common separatrix of two singular points.

In the smooth category one can remove a degeneracy of the foliation locally. Say, one can destroy a homoclinic loop by changing the foliation only in a flow-box around a point on the loop.

In the holomorphic category, a priori, one cannot perturb a foliation in a flow-box without changing the foliation globally. Our strategy to remove degeneracies in the holomorphic category is the following:

In Section 2 we construct a family of foliations, that eliminate degeneracy, in a neighborhood of a degenerate object, rather than in a flow-box around a point. A non-isolated cycle, a non-trivial pair of cycles are examples of degenerate objects. We give a complete list of degenerate objects in Section 2. All degenerate objects we consider are curves. Our technique allows us to construct an appropriate family only if a degenerate object is holomorphically convex. We expect though that it should be possible to carry out for any degenerate object.

In [7] our approach to construct a family of local foliations in a neighborhood of a degenerate object was to control the derivative of the holonomy map along the leaf with respect to a perturbation. This approach can not be adapted to remove a non-transversal intersection of strongly invariant manifolds. One cannot choose a leaf-wise path, that connects singular points with a point of non-transversal intersection. Therefore, one cannot control the intersection of invariant manifolds.

In this paper we use a different approach, a more geometric one. First, we reglue the neighborhood (Subsection 2.3). Then we project the obtained manifold, together with a new foliation, to the original one. We use Theorem [19], that states that a Stein manifold has a Stein neighborhood, to construct the projection.

We give a review of results on the holomorphic hulls of collections of curves in Section 3. We apply them to give geometric conditions for degenerate object to be holomorphically convex. We also review the relevant results from the Approximation Theory on Stein manifolds and apply them to pass from a local family of foliations in a neighborhood of a degenerate object to a global one.

When we remove a degenerate object, e.g. a complex cycle, we do not control the foliation outside a neighborhood of the degenerate object. Therefore, it might happen that eliminating one degenerate object we create many other in different places. We solve this problem as follows: We find a countable number of places where degenerate objects can be located. For each such location we prove that the complement to the set of foliations, which have the degenerate object at this particular location, is open and everywhere dense. Then we intersect these sets and get a residual set of foliations without holomorphically convex degenerate objects. We show that

if a foliation has a degenerate object, then it has a holomorphically convex degenerate object. Therefore, the residual set constructed does not have degenerate objects. We describe this strategy in detail in Section 4. This strategy was previously used in [7] and [11].

We give background information on Stein manifolds in the Appendix to make the paper accessible to the specialists, working in Dynamical Systems. There is also background information on holomorphic hulls and complex foliations in the Appendix. We also review facts on multiplicity of analytic sets, .

1.2 Acknowledgements

The author is grateful to Yulij Ilyashenko for the statement of the problem, numerous discussions and useful suggestions. We are also grateful to Igors Gorbovickis for proofreading an earlier version of the manuscript.

2 Local removal of degenerate objects.

2.1 List of degenerate objects.

As we pointed out in the introduction one can not eliminate a homoclinic saddle connection by changing the foliation only locally in a flow-box. Rather than that one needs to perturb the foliation in the neighborhood of the separatrix loop. This leads us to considering degenerate objects.

Below we list degenerate objects. One can check that if a foliation does not have degenerate objects of type 1-5, then it satisfies Theorem 1.1. If all singular points of a foliation are complex hyperbolic and it does not have degenerate objects of types 1 – 6, 8 – 9, then it is Kupka-Smale. If a all singular points of a foliation are complex hyperbolic and a foliation does not have degenerate objects of type 7, then it satisfies Theorem 1.3.

Definition 2.1. We say that γ is a degenerate object of a foliation \mathcal{F} if γ is

1. A non-trivial loop on a leaf L of \mathcal{F} , which is a representative of a non-hyperbolic cycle.
2. A union of loops γ_1, γ_2 that belong to the same leaf L of \mathcal{F} . We assume γ_1 and γ_2 are not multiples of the same cycle. Moreover, γ_1, γ_2 are hyperbolic. (See Fig. 1.)

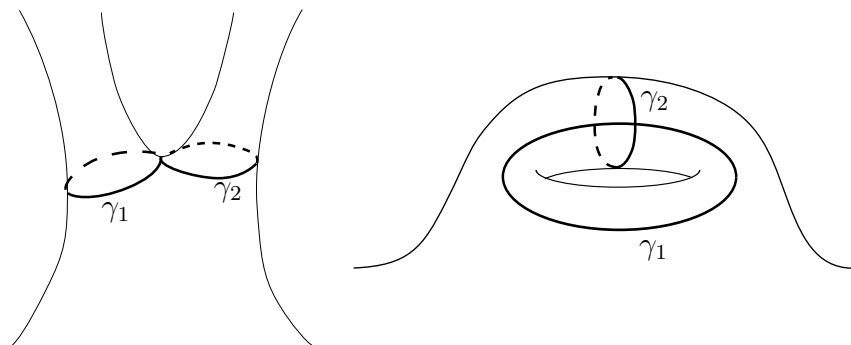


Figure 1: A pair of cycles

3. A path on a saddle connection, that connects two different hyperbolic singular points a_1 and a_2 . (See Fig. 2).
4. A loop on a homoclinic saddle connection S (See Fig.2):
 - a is a hyperbolic singular point;
 - S_1, S_2 are local separatrices of the singular point a ; $S_1 \neq S_2$; $S_1, S_2 \subset S$;
 - $\gamma \subset S$ passes through the singular point a , starts at S_1 , ends along S_2 .

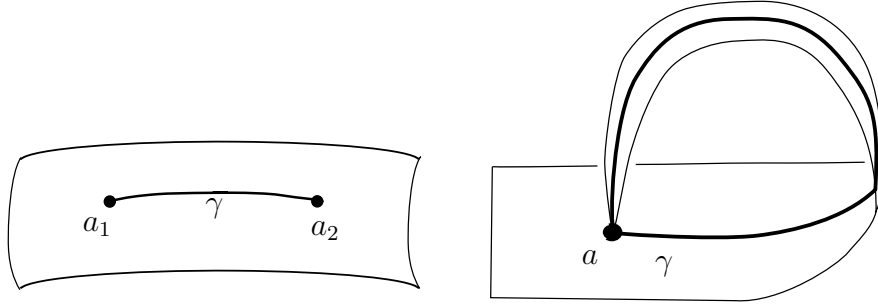


Figure 2: A path on a saddle connection and a loop on a homoclinic saddle connection

5. A non-trivial loop γ on a separatrix that passes through a singular point a (See Fig. 3).

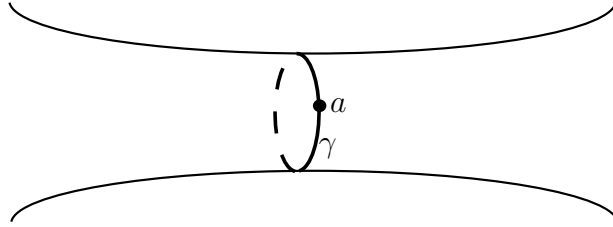


Figure 3: A loop on a separatrix.

6. A union of paths γ_1 and γ_2 (See Fig. 4):
 - a_1, a_2 are hyperbolic singular points of the foliation \mathcal{F} ;
 - M_1 and M_2 are strongly invariant manifolds of a_1 and a_2 correspondingly;
 - p is a point of non-transversal intersection of M_1 and M_2 ;
 - $\gamma_1 \subset M_1$ and $\gamma_2 \subset M_2$ are paths that connect a_1 and a_2 with the point p ;
 - $(\gamma_1 \cup \gamma_2) \setminus (M_1^{loc} \cup M_2^{loc}) \subset L$, where L is a leaf of foliation \mathcal{F} .
7. A loop $\gamma_1 \cup \gamma_2$ (See Fig. 5):
 - a is hyperbolic singular point of the foliation \mathcal{F} ;
 - M_1 and M_2 are strongly invariant manifold of the point a ;
 - $M_1^{loc} \cap M_2^{loc} = a$;
 - paths $\gamma_1 \subset M_1, \gamma_2 \subset M_2$ connect a with p ;

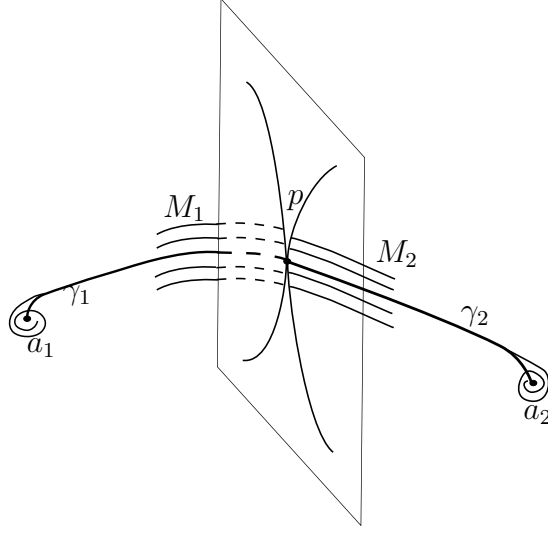


Figure 4: A non-transversal intersection of strongly invariant manifolds. The leaf L on the picture is not a separatrix, it spirals around singular points a_1 and a_2

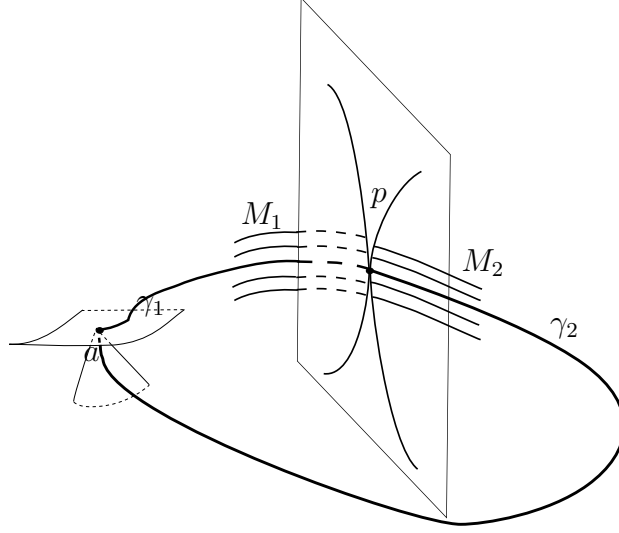


Figure 5: A homoclinic non-transversal intersection of strongly invariant manifolds.

- $(\gamma_1 \cup \gamma_2) \setminus (M_1^{loc} \cup M_2^{loc}) \cup L$, where L is a leaf of foliation \mathcal{F} .
8. A union $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:
- γ_1, γ_2 are hyperbolic loops on leaves of \mathcal{F} ;
 - M_1, M_2 are invariant manifolds of γ_1, γ_2 correspondingly;
 - $\gamma_3 \subset M_1, \gamma_4 \subset M_2$ are paths that connect points on γ_1, γ_2 with a point of non-transversal intersection of M_1, M_2 .
 - $(\gamma_3 \cup \gamma_4) \setminus (M_1^{loc} \cup M_2^{loc}) \subset L$, where L is a leaf of \mathcal{F} .
9. A union $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$:
- γ_1 is a hyperbolic loop on a leaf;

- M_1 is an invariant manifold of γ_1 ;
- a is a hyperbolic singular point;
- M_2 is a strongly invariant manifold of a ;
- $\gamma_2 \subset M_1$, $\gamma_3 \subset M_2$ are paths on invariant manifolds that connect a point on γ_1 and the point a correspondingly with the the point of non-transversal intersection of M_1 and M_2
- $(\gamma_2 \cup \gamma_3) \setminus (M_1^{loc} \cup M_2^{loc}) \subset L$, where L is a leaf of the foliation \mathcal{F} .

2.2 Local Removal Lemma.

In this section we find a neighborhood of a degenerate object and a family of holomorphic foliations in this neighborhood that eliminate the degenerate object in the neighborhood.

Our technique allows us to do that only if a degenerate object is holomorphically convex. We expect, though, it should be possible to carry out for any collection of smooth enough curve.

Let U be a neighborhood of the degenerate object. First, we allow not only the foliation, but the neighborhood itself to change with the parameter λ . We get a family of foliations \mathcal{F}_λ on manifolds U_λ . Then we find the way to 'project' U_λ to some neighborhood of the degenerate object. Thus, we produce a family of foliations in the neighborhood of the degenerate object that breaks it.

The following lemma summarizes the results of the following two subsections.

Let γ be a union of curves on a Stein manifold X , endowed with a foliation \mathcal{F}_0 . Assume it is holomorphically convex. Fix a point $p \in \gamma$, assume that $p \notin \Sigma(\mathcal{F})$. Assume that in a neighborhood of p the curve γ belongs to a leaf L of \mathcal{F}_0 . Let $\alpha \subset \gamma$ be a small arc, a neighborhood of p on γ . One can fix coordinates (z_1, \dots, z_{n-1}, t) in a neighborhood of the point p , so that t is a coordinate along the foliation. Consider the flow-box $\Pi = \{(z, t) : |z| < 1, t \in U(\alpha)\}$, where U_α is a neighborhood of an arc γ on the leaf L . Take a pair of points $q_1, q_2 \in \gamma \setminus \alpha$, that lie on different sides of α and belong to the flow-box Π . Let T_1, T_2 be transversal sections to \mathcal{F}_0 that pass through q_1, q_2 . Functions (z_1, \dots, z_{n-1}) work as coordinates on T_1, T_2 .

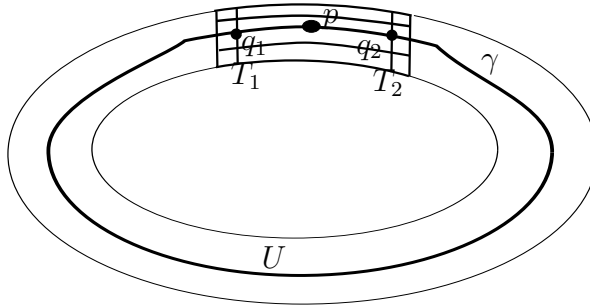


Figure 6: γ together with its neighborhood

Let Φ_λ be a holomorphic on λ family of germs of biholomorphisms

$$\Phi_\lambda : (\mathbb{C}^{n-1}, 0) \rightarrow (\mathbb{C}^{n-1}, 0), \quad \Phi_0 = Id.$$

Lemma 2.1. *There exist a neighborhood \tilde{U} of γ , that retracts to γ , such that $\Pi \cap \tilde{U} = \{(z, t) : |z| < \varepsilon, t \in U(\alpha)\}$, and a family of foliations \mathcal{F}_λ on \tilde{U} that depends holomorphically on λ satisfying the following conditions:*

1. in $\tilde{U} \setminus \Pi$, \mathcal{F}_λ is biholomorphic to \mathcal{F}_0 . More precisely, there exists a holomorphic on λ family of maps $\pi_\lambda : (\tilde{U} \setminus \Pi) \rightarrow X$, which are biholomorphisms to their images, such that π_λ maps the leaves of \mathcal{F}_0 to the leaves of \mathcal{F}_λ , $\pi_0 = \text{Id}$;
2. The holonomy map inside the flow-box along the foliation \mathcal{F}_λ between T_1 and T_2 is biholomorphically conjugate to Φ_λ , more precisely, in coordinates (z_1, \dots, z_{n-1}) on T_1, T_2 it is $(\pi_\lambda^z)^{-1} \circ \Phi_\lambda \circ \pi_\lambda^z$, where π_λ^z and $(\pi_\lambda^z)^{-1}$ are first $(n-1)$ coordinates of π_λ and π_λ^{-1} correspondingly.

This lemma mimics the smooth case, where one can perturb the foliation only in the flow-box. In the holomorphic case this is not possible. Therefore, we need to adjust everything by the map π_λ .

2.3 Regluing.

We weaken the restriction on curve γ for this section. We do not assume it to be holomorphically convex.

We start by constructing manifolds U_λ . They are obtained by regluing U in a flow-box around a point p . First, we describe the procedure informally and point out the technical difficulties that arise. Then we repeat the description paying attention to the technical difficulties.

We take a neighborhood U that can be retracted to γ . Let \hat{U} be a complex manifold obtained from U by doubling the preimage under retraction of a small arc, containing p . One can assume that the preimage of this small arc is a flow-box. So \hat{U} comes with the natural projection $\hat{U} \rightarrow U$, which is one-to-one everywhere except for the two flow-boxes around the preimages of p , which are glued together by the identity map. U_λ is obtained from \hat{U} by gluing the points in the flow-boxes by using the map $(\Phi_\lambda, \text{Id})$. The problem is that $(\Phi_\lambda, \text{Id})$ is not an isomorphism from the flow-box to itself. Thus, extra caution is needed to make U_λ Hausdorff. In the rest of the section we describe these precautions.

First, we choose a bigger neighborhood W that can be retracted to γ . Let ρ denote the retraction. Let \hat{W} be a connected complex manifold that projects one-to-one on $U \setminus \rho^{-1}(\alpha)$ and two-to-one on $\rho^{-1}(\alpha)$. Let π_1^{-1}, π_2^{-1} be the two inverses of the projection $\hat{W} \rightarrow W$, restricted to the preimage of $\rho^{-1}(\alpha) \subset W$.

Let V denote a flow-box around a point p in W . We assume $V \subset \rho^{-1}(\alpha)$. We take V small enough so that $(\Phi_\lambda, \text{Id})$ is a well-defined map on V and is a biholomorphism to its image. Let $V_1 = \pi_1^{-1}(V)$, $V_2 = \pi_2^{-1}(V)$.

Let $T_c \subset W$ be a tube of points that are at distance c from the preimage of $\gamma \setminus \alpha$. Let $\hat{T}_c \subset \hat{W}$ be a tube of points that are at the distance c from the preimage of $\gamma \setminus \alpha$. Take c small enough.

Take $U = T_c \cup V$, $\hat{U} = V_1 \cup V_2 \cup \hat{T}_c$. Note that U can be obtained from \hat{U} by gluing the points from V_1 and V_2 that project to the same point in W .

Let $V_2^\lambda = \pi_2^{-1}((\Phi_\lambda, \text{Id})(V))$

Let $U_\lambda = V_1 \cup \hat{T}_c \cup V_2^\lambda$. U_λ is a space obtained from \hat{U}_λ by gluing V_1 and V_2^λ by the map $(\Phi_\lambda, \text{Id})$. The space U_λ inherits complex structure. If one takes c and λ small enough, then it is also Hausdorff.

We also consider the total space of reglued manifolds:

$$\begin{aligned} \hat{\mathcal{U}} &= \{(u, \lambda) \in \hat{W} \times \Lambda \mid u \in V_1 \cup T_c \cup V_2^\lambda, \lambda \in \Lambda\} \\ \mathcal{U} &= \hat{\mathcal{U}} / \sim, (u, \lambda) \sim ((\Phi_\lambda, \text{Id})(u), \lambda), \text{ where } u \in V_1, \lambda \in \Lambda \end{aligned}$$

2.4 Projection. Siu's Theorem.

In this subsection we prove that for small enough λ one can take a small neighborhood of γ in U_λ and project it biholomorphically to a neighborhood of γ in U .

Assume γ is holomorphically convex. One can choose a neighborhood U_1 of γ , $U_1 \subset U$, such that U_1 is an analytic polyhedron, therefore, a Stein manifold ([8]).

By the theorem, formulated below there is a Stein neighborhood \tilde{U} of U_1 in \mathcal{U} .

Theorem 2.1. ([19]) *Suppose X is a complex space and A is a subvariety of X . If A is Stein, then there exists an open neighborhood Ω of A in X such that Ω is Stein.*

Fix an embedding of \tilde{U} into \mathbb{C}^N . We will need the following lemma:

Lemma 2.2. *There exists a linear $(N - n)$ -subspace $\alpha \subset \mathbb{C}^N$ such that the affine subspaces $\alpha_x \subset \mathbb{C}^N$ parallel to α passing through points $x \in \gamma$ are:*

- a) *transverse to U ;*
- b) *pass through only one point on γ .*

Proof: The set of all $(N - n)$ -subspaces of \mathbb{C}^N is $n(N - n)$ -dimensional complex manifold $Gr(N - n, N)$.

Elements of $Gr(N - n, N)$ that are not transverse to a given subspace of complementary dimension form a codimension 1 complex (singular) subvariety. Path γ is 1-dimensional real manifold. Therefore, subspaces that do not satisfy (a) form a subvariety of $Gr(N - n, N)$ of real codimension 1.

A couple of points on γ form a real 2-dimensional manifold. Linear subspace parallel to those that pass through two given points in \mathbb{C}^N form $(n(N - n - 1))$ -dimensional manifold. Therefore, subspaces that do not satisfy (b) form a submanifold of $Gr(N - n, N)$ of real codimension $2(n - 1)$.

Since $n \geq 2$, a $(N - n)$ -subspace α , that satisfies conditions (a) and (b), exists. \square

Proof of Lemma 2.1: Take α that satisfies Lemma 2.2. Let $\tilde{\pi}_\lambda$ be a projection along α from a neighborhood \tilde{U} of γ in U to U_λ , given by Lemma 2.2. One can take \tilde{U} be small enough, so that $\pi(\lambda) : \tilde{U} \rightarrow U_\lambda$ is a biholomorphism to its image for all small λ . Let $i_\lambda : U \setminus \rho^1(\alpha) \rightarrow U_\lambda$ be an identity map. It is easy to see that $\pi_\lambda = \tilde{\pi}_\lambda^{-1} \circ i_\lambda$ is a required map. \square

2.5 Removal of a holomorphically convex degenerate object.

A degenerate object is removed by a small perturbation if, roughly speaking, in some neighborhood of an object, there are no degenerate objects of the same kind for perturbed foliations.

Let γ be a degenerate object of a foliation \mathcal{F}_0 on a manifold X .

We say that \mathcal{F}_λ is a local holomorphic family for γ if there exists a neighborhood U of γ , such that \mathcal{F}_λ are well-defined in U for all $\lambda \in \Lambda$, $0 \in \Lambda$; and \mathcal{F}_λ depend holomorphically on λ .

Theorem 2.2. *Let γ be a holomorphically convex degenerate object of a foliation \mathcal{F}_0 . Then there exists a local holomorphic family of foliations \mathcal{F}_λ that removes γ .*

In the following subsection we rigorously define what it means that a degenerate object is removed in a local holomorphic family of foliations. We also prove Theorem 2.2 for different types of degenerate objects.

2.6 Removal of a non-hyperbolic cycle.

Definition 2.2. Let γ be a non-hyperbolic cycle of a foliation \mathcal{F}_0 . We say that it is removed in a local holomorphic family of foliations \mathcal{F}_λ if

1. there is a transversal section T at a point $p \in \gamma$ to the foliation \mathcal{F}_0 such that holonomy maps along γ for the foliation \mathcal{F}_λ , $\Delta_\gamma^\lambda : (D_r) \rightarrow T$ are well-defined for $\lambda \in \Lambda$, where $D_r \subset T$ is a disk of radius r with the center in a point p ;
2. for all $\lambda \in \Lambda \setminus R$, Δ_γ^λ has a unique fixed point on D_r , where R is a (possibly empty) one dimensional real-analytic set. Moreover, this fixed point is hyperbolic.

Proof of Theorem 2.2 for type 1: Take a point $p \in \gamma$ and a transversal section T to \mathcal{F} , $p \in T$. Let $\Delta_\gamma : (T, p) \rightarrow (T, p)$ be the corresponding holonomy map. The cycle γ is hyperbolic by the definition if and only if all the eigenvalues of Δ_γ lie not on the unit circle.

First, we provide a specific perturbation of Δ_γ that has hyperbolic fixed points only.

The following lemma is the standard fact:

Lemma 2.3. *There exists a diagonal $n \times n$ matrix D and $a \in \mathbb{C}^n$ such that the map $\Delta_\gamma(z) + \lambda(Dz + a)$ is well-defined and has hyperbolic fixed points only for all $\lambda \in V \setminus R$, where V is a neighborhood of 0, R is a (possibly empty) 1-dimensional real-analytic set, $0 \in R$.*

Take a, D such that Lemma 2.3 is satisfied.

Apply Lemma 2.1 to the cycle γ , the point p and the family of biholomorphisms $\Phi_\lambda = Id + \lambda(Dz + a)$. The map $\Delta_\gamma^\lambda = \pi_\lambda^{-1} \circ (\Delta_\gamma + \lambda(\Delta z + a)) \circ \pi_\lambda$ is the holonomy map along γ for the foliation \mathcal{F}_λ . For all λ outside a (possibly empty) one-dimensional real-analytic set R the map Δ_γ^λ has hyperbolic fixed points only on T . \square

2.7 Splitting cycles to different leaves

Let $\gamma = \gamma_1 \cup \gamma_2$ be a degenerate object of type 2.

Definition 2.3. We say that γ is removed in a holomorphic family of foliations \mathcal{F}_λ , $\lambda \in \Lambda$ if

1. there is a transversal section T at a point $p \in \gamma_1 \cap \gamma_2$ to the foliation \mathcal{F}_0 such that holonomy maps $\Delta_{\gamma_1}^\lambda, \Delta_{\gamma_2}^\lambda : D_r \rightarrow T$ are well-defined for all $\lambda \in \Lambda$, where $D_r \subset T$ is a disk of radius r ;
2. $\Delta_{\gamma_1}^\lambda$ and $\Delta_{\gamma_2}^\lambda$ do not have a common fixed point on D_r for $\lambda \neq 0$.

Thus, the degenerate object is removed if γ_1 and γ_2 split to leaves, that are different at least in U .

Proof of Theorem 2.2 for type 2: Let $q \in \gamma_1 \setminus \gamma_2$. Assume, it is not a point of self-intersection of γ_1 . Apply Lemma 2.1 to the curve γ , the point q , and the family of biholomorphisms $\Phi_\lambda = z + \lambda$. Then $\pi_\lambda^{-1} \circ \Delta_{\gamma_2} \circ \pi_\lambda$ is a holonomy map along γ_2 . Let T_1 be a transversal section to the foliation \mathcal{F}_0 in a point q . The holonomy map along γ_1 for the foliation \mathcal{F}_0 can be written as a composition $\Delta_{\gamma_1} = \Delta_2 \circ \Delta_1$, where Δ_1 is a holonomy map from transversal section T to T_1 , Δ_2 is a holonomy map from T_1 to T . Then the holonomy map along γ_1 for the foliation \mathcal{F}_λ is $\pi_\lambda^{-1} \circ \Delta_2 \circ \Phi_\lambda \circ \Delta_1 \circ \pi_\lambda$.

$\pi_\lambda^{-1}(p)$ is an isolated fixed point for the holonomy map along γ_2 and is not a fixed point for the holonomy map along γ_1 . Thus, cycles split to different leaves. \square

2.8 Removal of non-transversal intersections of invariant manifolds and saddle connections.

Let γ be a degenerate object of the foliation \mathcal{F}_0 of types 3 – 9. Let \mathcal{F}_λ be a family of local foliations for γ .

In the sequel the words “invariant manifold” stand for a strongly invariant manifold, or separatrix of a singular point; or stable, unstable manifolds of a complex cycle. These objects persist under the perturbation, and depend holomorphically on a foliation. The local strongly invariant manifolds and separatrices of a_i and local stable/instable manifolds of γ_i persist under the perturbation and depend holomorphically on λ .

For each degenerate object of type 3 – 9, there are two invariant manifolds that meet nontransversally. Saddle connections are examples of non-transversal intersection. We denote the corresponding local invariant manifolds by M_1^{loc} and M_2^{loc} for the foliation \mathcal{F}_0 ; $M_1^{loc}(\lambda)$ and $M_2^{loc}(\lambda)$ for the foliation \mathcal{F}_λ . Note that for the degeneracy of type 5, $M_1^{loc} = M_2^{loc}$.

For degenerate objects of types 3 – 5, we can take p to be any point in $\gamma \setminus (M_1^{loc} \cup M_2^{loc})$.

Notice that for all degenerate objects of type 3 – 9, $\gamma \setminus (M_1^{loc} \cup M_2^{loc}) \subset L$, where L is a leaf of foliation \mathcal{F}_0 . Therefore, holomorphic extensions $M_1(\lambda)$ and $M_2(\lambda)$ of $M_1^{loc}(\lambda)$ and $M_2^{loc}(\lambda)$ along γ are well-defined.

Definition 2.4. We say that γ can be eliminated in a holomorphic family of foliations \mathcal{F}_λ if there exists a transversal section T to the foliation \mathcal{F}_0 , $p \in T$, so that $M_1(\lambda)$ and $M_2(\lambda)$ intersect transversally on T .

Note 2.1. Note that if M_1^{loc} and M_2^{loc} are separatrices, then the holomorphic family eliminates the saddle connection.

Proof of Theorem 2.2 for types 3-9. One can assume that in a neighborhood of a point p M_1 and M_2 are biholomorphically equivalent to $m_1 \times D$, $m_2 \times D$, where $m_1 = M_1 \cap D_1$, $m_2 = M_2 \cap D_1$, D is a neighborhood of p on the leaf L ; D_1 is a neighborhood of p on the transversal section T . Fix coordinates (z_1, \dots, z_{n-1}) on T . Apply Lemma 2.1 to the curve γ , the point p and $\Phi_\lambda = z + \lambda a$. Assume that points $q_1 \in \gamma_1$, $q_2 \in \gamma_2$.

Outside of the flow-box $M_1(\lambda) = \pi_\lambda(M_1)$, $M_2(\lambda) = \pi_\lambda(M_2)$.

In a neighborhood of the point p :

$$T \cap M_2(\lambda) = \pi_\lambda^z(m_2)$$

$$T \cap M_1(\lambda) = \Phi_\lambda \circ \pi_\lambda^z(m_1).$$

Therefore, by Sard’s Theorem, for almost all a they intersect transversally. \square

3 Construction of a global eliminating family.

In this section we give the geometric conditions for degenerate objects to be holomorphically convex and show how to pass from a local foliation to a global one.

3.1 Approximation Theory.

Working in the category of smooth vector fields one can eliminate a non-transversality by perturbing the vector field only in a neighborhood of a non-transversality. In the holomorphic

category there are no local perturbations allowed. However, approximation theory gives a way to work locally. In some cases you can perturb the local picture and then approximate your perturbation by a global one. In particular, for a holomorphic vector bundle on a Stein manifold holomorphic sections over a neighborhood of a holomorphically convex set can be approximated by global holomorphic sections. This follows from two theorems formulated below.

Theorem 3.1. ([12], 5.6.2) *Let X be a Stein manifold and φ a strictly plurisubharmonic function in X such that $K_c = \{z: z \in X, \varphi(z) \leq c\} \Subset X$ for every real number c . Let B be an analytic vector bundle over X . Every analytic section of B over a neighborhood of K_c can then be uniformly approximated on K_c by global analytic sections of B .*

Theorem 3.2. ([12], 5.1.6) *Let X be a Stein manifold, K a compact subset of X and U is an open neighborhood of holomorphic hull of K . Then there exists a function $\varphi \in C^\infty(X)$ such that*

1. φ is strictly plurisubharmonic,
2. $\varphi < 0$ in K but $\varphi > 0$ in $X \setminus U$,
3. $\{z: z \in X, \varphi(z) < c\} \Subset X$ for every $c \in \mathbb{R}$.

Theorem 3.3. *Let γ be a holomorphically convex degenerate object of a foliation \mathcal{F}_0 . Then there exists a holomorphic family \mathcal{F}_λ of foliations on X , that remove γ .*

Proof: Let s_λ be local sections that determine local foliations that eliminate γ . Let $\lambda_0 \in \Lambda$ be a parameter that does not belong to exceptional real curve. By Theorems 3.1 and 3.2 there exists a global section S_{λ_0} that is ε -close to s_{λ_0} on U' , where $\gamma \Subset U' \Subset U$. Therefore, family of foliations determined by $S_\lambda = S_0 + \lambda(S_{\lambda_0} - S_0)$ eliminate the degenerate object. \square

3.2 Holomorphic convexity of a curve.

We recall the definition of a holomorphic hull and gave examples of holomorphic hulls of curves in Appendix 5.1.

Consider a collection of C^1 - smooth real curves $\gamma_1, \dots, \gamma_m$ in \mathbb{C}^N . Their holomorphic hull is described by Stolzenberg's Theorem [20]:

Theorem 3.4. *Let $\gamma = \gamma_1 \cup \dots \cup \gamma_m$. Then $h(\gamma) \setminus \gamma$ is a (possibly empty) one-dimensional analytic subset of $\mathbb{C}^N \setminus \gamma$.*

Corollary 3.1. *The statement of the theorem is true if one replaces \mathbb{C}^n by a Stein manifold.*

Proof: There exists a proper embedding of the Stein manifold X to \mathbb{C}^N for some large enough N ([12], theorem 5.3.9). Let $h(\gamma)$ be the holomorphic hull of γ in \mathbb{C}^N . By Stolzenberg's theorem $h(\gamma) \setminus \gamma$ is an analytic subset in $\mathbb{C}^N \setminus \gamma$.

Let us show that $h(\gamma) \subset X$. X is a maximal spectrum of functions that are equal to zero on X ([8], Theorem VII, A18). Take a function f such that $f(X) = 0$. Then $f(h(\gamma)) = 0$ since $\gamma \subset X$ and $h(\gamma)$ is a holomorphic hull of γ in \mathbb{C}^N . Thus, $h(\gamma) \subset X$.

It remains to show that $h(\gamma) = h_X(\gamma)$. Any holomorphic function on X is a restriction of holomorphic function on \mathbb{C}^N ([8], Theorem VII, A18). Therefore, $h_X(\gamma) = h(\gamma) \cap X$. Since $h(\gamma) \subset X$, $h_X(\gamma) = h(\gamma)$. \square

3.3 Holomorphic convexity of a degenerate object.

In this subsection we give the geometric conditions for the degenerate objects to be holomorphically convex.

Definition 3.1. We say that a path or a loop is *simple* if it does not have points of self-intersection.

We need to extend the analytic set, given by Stolzenberg's theorem, to the boundary. In the sequel we need the following corollary from the Stolzenberg's Theorem.

Corollary 3.2. *Let $\gamma_1, \dots, \gamma_n$ be piece-wise smooth curves, such that $\gamma_i \cap \gamma_j$ consists of finite number of points. Suppose that $h(\gamma) \not\subset \gamma$. Then there exists an arc $\alpha \subset \gamma_i$, such that $\alpha \subset \partial(h(\gamma) \setminus \gamma)$, where $\gamma = \cup \gamma_i$.*

Proof: Let π be a projection of $\gamma_1, \dots, \gamma_n$ to a complex line C . One can choose π so that image of γ has at most finite number of points of transversal self-intersection. The image of γ separate C into several regions U_i .

Below we show that for each of the regions there is a following dichotomy: either $\pi(h(\gamma)) \supset U_i$ or $\pi(h(\gamma)) \cap \text{int}(U_i) = \emptyset$:

First, $\pi(h(\gamma) \setminus \gamma)$ is open. Therefore, the $\pi(h(\gamma) \setminus \gamma) \cap U_i$ is open. Second, if $w \in \partial \pi(h(\gamma))$, then since $h(\gamma)$ is compact, there exists $z \in h(\gamma)$ such that $\pi(z) = w$. Therefore, $z \in \gamma$. Thus, $\pi(h(\gamma) \setminus \gamma) \cap U_i$ is either empty or coincides with U_i .

Take a point $w \in \pi(\gamma)$ that is not a point of self-intersection and belongs to the boundary of $\pi(h(\gamma))$. The small arc around this point also belongs to $\pi(h(\gamma))$ and does not contain points of self-intersection. The preimage of this arc is the desired arc on γ . \square

Theorem 3.5 ([5]). *Let M be a connected $(2p-1)$ -dimensional C^1 -submanifold of a complex manifold Ω . Let A_1, A_2 be irreducible p -dimensional analytic subsets of $\Omega \setminus M$ such that the closure of each of them contains M . Then either $A_1 = A_2$ or $A_1 \cup M \cup A_2$ is an analytic subset of Ω .*

Lemma 3.1. *Let $\alpha \subset \gamma$ be a real-analytic arc, such that $\alpha \subset \partial(h(\gamma) \setminus \gamma)$. Let C be a holomorphic curve, such that $\alpha \subset C$. Then there exists a loop $\tilde{\gamma} \subset \gamma$, so that $\alpha \subset \tilde{\gamma} \subset \gamma \cap C$ and $\tilde{\gamma}$ is null homologous on C .*

Proof: One can take a neighborhood $U \subset X$ of the arc α , such that

1. $U \cap \gamma = \alpha$;
2. the connected component of $C \cap U$, that contains α , is a submanifold in U ;
3. the arc α separates this connected component into two pieces. Let Ω_1, Ω_2 be these pieces.

Let h_1 denote the connected component of $h(\gamma) \setminus \gamma$.

Apply Theorem 3.5 to the analytic sets h_1 and Ω_1 and the arc α . The closure of h_1 in U contains α . The closure of Ω_1 also contains α . Therefore, either $h_1 = \Omega_1$ or $h_1 \cup \alpha \cup \Omega_1$ is an analytic subset of U . In the second case $h_1 = \Omega_2$. Thus, $h_1 = \Omega_1$ or $h_1 = \Omega_2$. If two analytic sets coincide locally, then they coincide globally. Therefore, $h_1 \subset C$.

By Maximum Modulus Principle, $\partial h(\gamma) \subset \gamma$. Denote $\tilde{\gamma} = \partial h_1$, then it is a loop and is null-homologous on C . \square

Theorem 3.6. *Let γ be a degenerate object of the foliation \mathcal{F} . When $\gamma = \cup \gamma_i$, then we assume γ_i are simple piece-wise real-analytic. Suppose γ satisfies the following additional conditions:*

type 1 : Let γ be non-homologous to 0 on the leaf L .

type 2: (a) γ_1 and γ_2 have only one common point;

(b) γ_1 and γ_2 are not null homologous and are not multiples of the same cycle in the homology group of L .

type 5: γ is not null-homologous on S .

type 8: $\gamma_1 \subset L$ is non-homologous to 0 on L ; γ_1 and γ_2 have only one common point.

type 9: $\gamma_1 \subset L_1$, $\gamma_2 \subset L_2$ are non-homologous to zero on L_1, L_2 correspondingly, $L_1 \neq L_2$. Curves γ_1 and γ_3 ; γ_2 and γ_4 have only one common point.

Then γ is holomorphically convex.

Note 3.1. If γ satisfies the listed above geometric conditions, then we say that γ is a *geometric degenerate object*.

Proof: Suppose γ is not holomorphically convex.

type 1: Since γ is a simple cycle, by Lemma 3.1, γ is null-homologous on L , which contradicts the hypothesis.

type 2: The proof is the same as for type 1.

type 3: γ is simply connected, therefore, by Lemma 3.1 it should bound the region on S , which contradicts the hypothesis.

type 4: Let \tilde{S} be a surface obtained from $S \cup \{a\}$ by splitting the local components of S at the point a . Let $\pi : \tilde{S} \rightarrow S$ be the corresponding projection.

$\pi^{-1}(\gamma)$ is a simple path on \tilde{S} . It does not bound a region on \tilde{S} . Therefore, its image does not bound a region on S . Which contradicts Lemma 3.1.

type 5: The same as for type 1.

type 6: Let α be an arc, given by Corollary 3.2. Suppose $\alpha \subset \gamma_1$. Let C be a curve, given by 3.1. Then C is either a saddle connection or $C \subset M_1$. Saddle connections are removed. Therefore, by Lemma 3.1 γ_1 bounds a region on C . Which contradicts the hypothesis.

type 7: The proof is the same as for type 6.

type 8: Let α be an arc, given by Corollary 3.2. Suppose $\alpha \subset \gamma_1$, then by Lemma 3.1, γ_1 is null-homologous on L_1 , which contradicts the hypothesis. If $\alpha \subset \gamma_3$, then we proceed by the same reasoning as for type 6.

type 9: The proof is the same as for type 8.

□

4 Simultaneous elimination of degeneracies.

4.1 Landis-Petrovskii's lemma.

The idea is to encode degeneracies by countably many objects. To give a feeling of the method used, we first prove a version of the Landis-Petrovskii's Lemma [16] that we need in the sequel.

Lemma 4.1. *For a holomorphic 1-dimensional (singular) foliation \mathcal{F} of a Stein manifold X there exists not more than a countable number of isolated complex cycles on the leaves of the foliation.*

Proof: Since the manifold X is Stein, it can be embedded into \mathbb{C}^N . Take a cycle γ on a leaf L .

Fix coordinates (z_1, \dots, z_N) in \mathbb{C}^N . Let C_1, \dots, C_N be the coordinate lines,

$$C_i = \{z_1, \dots, z_N = 0\}.$$

Suppose that L does not belong to the hypersurface $\{z_i = c\}$ for any $c \in \mathbb{C}$. By perturbing γ on the leaf L one can assume that there exists a small neighborhood $U \supset \gamma$ so that $\pi_i|_U$ is a biholomorphism to the image (here $\pi_i : \mathbb{C}^N \rightarrow C_i$, $\pi_i(z) = z_i$ is the projection). Then one can perturb γ inside U so that $\pi_i(\gamma)$ becomes a piece-wise linear curve with rational vertices.

Definition 4.1. We will say that the cycle γ' *lies over* the piece-wise linear curve g' if there exist a representative of γ' and its neighborhood U' , such that U' is projected biholomorphically to its image and the representative is projected to g' . Note, that any cycle lies over countably many piece-wise linear curves.

Take one of the vertices of $\pi_i(\gamma)$, say with coordinate $z_i = c$. The hypersurface $\{z_i = c\}$ intersects X by $(k-1)$ -dimensional variety, such that for any cycle γ' , lying over $\pi_i(\gamma)$, it is transversal to the foliation in a neighborhood of $\gamma' \cap \{z_i = c\}$. The holonomy map along γ is well-defined in some neighborhood of the intersection $\{z_i = c\} \cap \gamma$. The holonomy map does not have any other fixed points in some smaller neighborhood. Thus, each cycle that projects to the same piece-wise linear curve gives a neighborhood on the hyperplane $\{z_i = c\} \subset \mathbb{C}^N$, so that two neighborhoods for two different cycles do not intersect each other. Therefore, there exists not more than countably many limit cycles that project to the same curve. Since there are only countably many curves, there are not more than countably many limit cycles. \square

Landis-Petrovskii's Lemma implies that once all non-isolated cycles are eliminated, all leaves except for countably many are homeomorphic to disks.

4.2 Simultaneous elimination of non-isolated cycles.

If there are non-isolated cycles on the leaves of a foliation \mathcal{F} , then the number of the cycles is obviously uncountable. However, the strategy described above can be applied. Our idea is to catch the degenerations by a countable number of holonomy maps.

Theorem 4.1. *There exists a residual set \mathcal{R}_1 in the space of 1-dimensional singular holomorphic foliations, that do not have geometric degenerate objects of type 1.*

Proof: Since X is Stein, it can be embedded into \mathbb{C}^N . We can restrict ourselves to the foliations without leaves that belong to the hypersurfaces $\{z_N = c\}$, $c \in \mathbb{C}$. The set of such foliations is open and dense. We describe the holonomy maps that catch all the cycles for all foliations.

We introduce the following notations:

- \mathcal{A} is a countable, everywhere dense subset in the set of holomorphic foliations;
- \mathcal{G} is the set of all closed piecewise-linear curves with rational vertices on

$$\{z_1 = \dots = z_{N-1} = 0\},$$

with one marked vertex.

- Let $\tau_q = \{z_n = q\} \cap X$, where $q \in \mathbb{Q} + i\mathbb{Q}$.

Let \mathcal{Q}_q be a countable everywhere dense set on τ_q .

Let $\mathcal{Q} = \bigsqcup \mathcal{Q}_q$.

Let $\mathbf{z} = (z_1, \dots, z_{N-1})$, $u = z_N$.

Consider a 4-tuple $\alpha = (\mathcal{F}, g, \mathbf{z}, r) \in (\mathcal{A}, \mathcal{G}, \mathcal{Q}_q, \mathbb{Q})$, such that q is the marked point of g . We require that the holonomy map for the foliation \mathcal{F} in the point \mathbf{z} along g is well-defined in a neighborhood of \mathbf{z} on the transversal section τ_q and has the radius of convergence greater than r . Let Δ_α be the germ of this holonomy. One can consider the germ of the holonomy map along the lifting of g , starting at \mathbf{z} , for foliations close to \mathcal{F} . Therefore, we think of Δ_α as of function of two variables: a foliation close to \mathcal{F} , and a point on the transversal section τ_q .

Below we fix a specific representative of Δ_α . We use the same notation for the specific representative as for the germ.

Let V_α be the connected component, containing \mathcal{F} , of the set of foliations, for which the holonomy map along g in the point \mathbf{z} is well-defined and has radius of convergence greater than r . The domain of definition of Δ_α is

$$\{(\mathcal{F}', \mathbf{z}') \mid \mathcal{F}' \in V_\alpha, |\mathbf{z}' - \mathbf{z}| < r\}.$$

Note, that V_α is open.

From this point we consider fixed representatives, rather than germs.

Lemma 4.2. *Every complex cycle corresponds to a fixed point of $\Delta_\alpha(\mathcal{F}', \cdot)$ for some α and $\mathcal{F}' \in V_\alpha$.*

Proof: Let γ be a complex cycle on a leaf L of a foliation \mathcal{F} . One can perturb γ on L so that it projects to some $g \in \mathcal{G}$. Let $u(g)$ be one of the vertices of the projection, and let $\mathbf{z} \in \gamma$ be the preimage of $u(g)$. Consider the holonomy map along γ in a neighborhood of \mathbf{z} in the transversal section $C = \{u = u(g)\}$. Take a point $\mathbf{z}_1 \in \mathcal{Q}$ such that $|\mathbf{z} - \mathbf{z}_1| < r_{\mathbf{z}}(\mathcal{F})/4$ where $r_{\mathbf{z}}(\mathcal{F})$ is a radius of convergence of the holonomy map in the point \mathbf{z} along γ for the foliation \mathcal{F} . Note, that $r_{\mathbf{z}_1}(\mathcal{F}) > r_{\mathbf{z}}(\mathcal{F})/2$. One can take \mathcal{F}_1 close to \mathcal{F} so that $r_{\mathbf{z}_1}(\mathcal{F}_1) > r_{\mathbf{z}}(\mathcal{F})/2$. Denote by $\alpha = (\mathcal{F}_1, g, \mathbf{z}_1, r)$, where $r \in \mathbb{Q}$, $r_{\mathbf{z}}(\mathcal{F})/4 < r < r_{\mathbf{z}}(\mathcal{F})/2$. Then $r < r_{\mathbf{z}_1}(\mathcal{F}_1)$. Also, $\mathcal{F} \in V_\alpha$, because $r_{\mathbf{z}_1}(\mathcal{F}) > r$. Since $r > r_{\mathbf{z}}(\mathcal{F})/4$, the point z belongs to the domain of definition of $\Delta_\alpha(\mathcal{F}_1, \cdot)$. \square

Lemma 4.3. *Fix Δ_α . The set $D_\alpha \subset V_\alpha$ of foliations \mathcal{F} such that $\Delta_\alpha(\mathcal{F}, \cdot)$ has a non-hyperbolic fixed point, so that the corresponding cycle γ satisfies additional conditions:*

1. γ is simple,
2. γ is null homologous on the leaf;

is closed and nowhere dense in V_α .

Proof: We prove that by a finite number of steps, we can perturb the foliation \mathcal{F} so that $\Delta_\alpha(\tilde{\mathcal{F}}, \cdot)$ has isolated fixed points only in the domain of definition discussed above. Assume that A is the set of fixed points of $\Delta_\alpha(\mathcal{F}, \cdot)$. Let A be k -dimensional. As we show in the appendix, one can associate multiplicity $m(A)$ to the analytic set A . Take a point z that is a generic point of a k -dimensional stratum A_i . By Theorem 3.3, there exists a neighborhood of z and a foliation $\tilde{\mathcal{F}}$, arbitrary close to \mathcal{F} , such that the holonomy map of $\tilde{\mathcal{F}}$ along γ has isolated fixed points only in this neighborhood.

This perturbation destroys the component A_i . Therefore, it either decreases the dimension of A , or it decreases the multiplicity $m(A)$ (see Lemma 5.4).

Therefore, after a finite number of steps, only isolated cycles are left. By the Theorem 3.3, they can be turned into hyperbolic by a finite number of steps as well. \square

Corollary 4.1. *The complement of D_α in the set of all foliations contains an open everywhere dense set.*

The residual set is obtained by intersecting open everywhere dense sets from the corollary above. \square

4.3 Simultaneous splitting of cycles to different leaves.

Theorem 4.2. *There exists a residual set in the space of singular holomorphic 1-dimensional foliations that do not have geometric degenerate objects of type 2.*

Proof: The construction is similar to Section 4.2. The difference is that one needs to consider pairs of holonomy maps and the analytic condition is that they do not have a common fixed point. \square

4.4 Simultaneous elimination of separatrices and non-transversal intersections of invariant manifolds

Theorem 4.3. *There exists a residual set in the space of singular holomorphic 1-dimensional foliations that do not have geometric degenerate objects of types 3 – 9.*

Proof: We outline the proof for strongly invariant manifolds of different singular points. For other types of degenerate objects the proof goes along the same lines.

Since X is a Stein manifold, it can be embedded into \mathbb{C}^N .

We fix the countable set of data $\alpha = (\mathcal{F}, a_1, M_1, a_2, M_2, g, z_1, r)$.

- $\mathcal{F} \in \mathcal{A}$, where \mathcal{A} is a countable every-where dense set of foliations;

Foliations with hyperbolic singular points only form a residual set [4]. Therefore, we can assume that all singular point for all the foliations $\mathcal{F} \in \mathcal{A}$ are hyperbolic.

- a_1, a_2 are hyperbolic singular points of \mathcal{F} ;

- M_1, M_2 are strongly invariant manifolds of a_1 and a_2 correspondingly;

We associate the maximal radius r_i to the singular point a_i .

Definition 4.2. The radius r_i is the maximal radius, such that M_i is transversal to $\partial U_r(a_i)$ for all $r < r_i$.

Not that maximal radius is a lower semicontinuous function on the space of foliations.

Let $\pi : X \rightarrow C$ be the projection to $C = \{z_1 = \dots = z_{N-1} = 0\}$, $\pi(x_1, \dots, x_N) = x_N$.

- $g \subset C$ is a piecewise linear curve with rational vertices. Let u_1, u_2 be the starting and the ending points of g correspondingly. We require that $u_1 \in \pi(U_{r_1}(a_1))$, $u_2 \in \pi(U_{r_2}(a_2))$;
- $z_1 \in \mathcal{Q}_q$, where \mathcal{Q}_q is an every where dense set on the transversal section $\tau_1 = \{z_n = u_1 = q\} \cap X$ in $U_{r_1}(a_1)$;

We require that there is a well-defined lift of g to the leaf L of the foliation \mathcal{F} , that starts from a point z_1 . The lift is denoted by γ . Let z_2 be the lift u_2 . We require that $z_2 \in U_{r_2}(a_2)$

Let $\tau_2 = \{z_N = u_2\} \cap X$.

There is a well-defined germ $\Delta : \tau_1 \rightarrow \tau_2$ of the holonomy map along γ in the point z_1 .

As before, we think of Δ as a function of two variables: a foliation \mathcal{G} , close to \mathcal{F} , and a point on the transversal section τ_1 .

- $r \in \mathcal{Q}_+$. We require that
 1. r is less than radius of convergence of Δ .
 2. The disk $D_r(z_1)$ on the transversal section τ_1 of the radius r_1 with the center z_1 is compactly contained in $U_{r_1}(a_1)$.
 3. $\Delta(D_r(z_1))$ is compactly contained in $U_{r_2}(a_2)$.

We fix a representative Δ_α of Δ . Below we describe the neighborhood U_α of \mathcal{F} . \mathcal{G} belongs to U_α if

1. there is a holomorphic family of foliations \mathcal{F}_λ , so that $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_1 = \mathcal{G}$; for all $\lambda \in D_1$ there are unique hyperbolic singular points $a_1^\lambda \in U_{r_1/2}(a_1)$ and $a_2^\lambda \in U_{r_2/2}(a_2)$ of the foliation \mathcal{F}_λ ;

Let a'_1, a'_2 be singular points of \mathcal{G} , obtained via holomorphic continuation. Let M'_1, M'_2 be the corresponding strongly invariant manifolds. Let r'_1, r'_2 be the maximal radii for $(a'_1, M'_1), (a'_2, M'_2)$.

2. $z_1 \in U_{r'_1}(a'_1)$, $z'_2 \in U_{r'_2}(a'_2)$, where z'_2 is the lift of u_2 along g for \mathcal{G} .
3. $D_r(z_1)$ is compactly contained in $U_{r'_1}(a'_1)$.
4. $\Delta(\mathcal{G}, D_r(z_1))$ is compactly contained in $U_{r'_2}(a'_2)$.

The domain of definition of Δ_α is $U_\alpha \times D_r(z_1)$.

Lemma 4.4. For any α , the set $D_\alpha \subset U_\alpha$ of foliations $\mathcal{G} \subset U_\alpha$, for which there exists a leaf L such that

1. the lift of u_1 to L is in $U_{r_1}(a_1)$, the lift of u_2 to L is in $U_{r_2}(a_2)$;
2. the lift of g belongs to the strongly invariant manifold M'_1 of the singular point a'_1 of \mathcal{G} (a'_1 is a holomorphic continuation of a_1);
3. the lift of u_2 belongs to the strongly invariant manifold M'_2 of a singular point a'_2 (a'_2 is a holomorphic continuation of a_2);
4. the lift of u_2 is a point of a non-transversal intersection of M'_1 and M'_2 .

is a closed and nowhere dense set.

Proof: The proof follows from the local Theorem 3.3 in the same way as in Lemma 4.3. \square

The desired residual set is obtained by intersecting the open everywhere dense sets from the above Corollary. \square

4.5 Proofs of main theorems.

Theorem 4.4. *A foliation \mathcal{F} , that does not have geometric degenerate objects of types 1 – 5 satisfies Theorem 1.1.*

Proof: If a leaf L is not contractible, then there exists a simple loop $\gamma \subset L$, non-homologous to zero on L . Foliation \mathcal{F} does not have geometric non-isolated cycles. Hence, if L is a non-contractible leaf of a foliation \mathcal{F} , then there is a geometric isolated cycle $\gamma \subset L$. By Landis-Petrovskii's Lemma (Section 4.1), there are at most countably many isolated cycles. Thus, all leaves, except for countably many, are contractible.

If $H^1(L, \mathbb{Z}) \neq 0, \mathbb{Z}$, then there exist a pair of cycles $\gamma_1, \gamma_2 \subset L$, that satisfy geometric conditions. Since foliation \mathcal{F} does not have a pair of geometric cycles, all non-separatrix leaves L are either contractible or $H^1(L, \mathbb{Z}) = \mathbb{Z}$.

Since foliation \mathcal{F} does not have geometric degenerate objects of types 3-5, one can show the same way, that separatrix leaves are topological cylinders. \square

Theorem 1.1 is an immediate corollary of Theorems 4.4, 4.1, 4.2, 4.3.

Theorem 4.5. *If a foliation \mathcal{F} does not have geometric degenerate objects of types 1 – 6, 8 – 9, then it is complex Kupka-Smale.*

Proof: By Theorem 4.4 all leaves of foliation \mathcal{F} are either contractible or cylinders. Since the foliation does not have geometric non-hyperbolic cycles, all cycles are hyperbolic.

Suppose there is a non-transversal intersection of invariant manifolds M_1, M_2 . Let p be a point of non-transversal intersection. Let L be a leaf of foliation, such that $p \in L$. Since $L \subset M_1$, there is a path $\gamma_1 \subset L$ that connects p with a point $q \in M_1^{loc}$, one can assume that γ_1 is simple and piece-wise real analytic. The same way, there is a leaf-wise γ_2 path from p to M_2^{loc} . Thus we constructed a geometric degenerate object of type 6 – 9, which contradicts the hypothesis. \square

Thus Theorem 1.2 is an immediate consequence of Theorem 1.1 and Theorem 4.3.

The same way one shows that Theorem 1.3 is a corollary from Theorem 4.3.

5 Appendix

5.1 Stein manifolds.

In this subsection we state the well-known facts about Stein manifolds. For the proofs and further discussion, consult [12].

Whitney Embedding Theorem states that any smooth m -dimensional manifold can be smoothly embedded into Euclidean $2m$ -space. For complex holomorphic manifolds the situation is different. There are complex manifolds that cannot be holomorphically embedded as submanifolds to \mathbb{C}^n . Moreover, there are ones that do not admit any global holomorphic functions, except for constants.

By Maximum Modulus Theorem and Liouville's Theorem compact manifolds do not admit any nonconstant global holomorphic functions.

Informally speaking, Stein manifolds are the ones which do have an ample supply of holomorphic functions.

We start our discussion of Stein manifolds with the definition of the holomorphic hull. This notion plays an important role in the theory.

Definition 5.1. Let K be a compact subset of a complex manifold X , the $\mathcal{O}(X)$ -hull of K is the set

$$h_X(K) = \{u : |f(u)| \leq \max\{|f(x)| : x \in K\} \text{ for all } f \in \mathcal{O}(X)\},$$

where $\mathcal{O}(X)$ are holomorphic functions on X .

Note 5.1. We also call $\mathcal{O}(X)$ -hull, the holomorphic hull, when it is clear from the context what the ambient manifold is. The notation $h(K)$ is used in that case.

Note 5.2. The holomorphic hull is a reasonable notion, only if the manifold has an ample supply of holomorphic functions. For instance, it is an important notion for the compact subsets of \mathbb{C}^n .

Example 5.1. The holomorphic hull of the curve $\{|z| = 1\} \subset \mathbb{C}$ is $\{|z| \leq 1\}$, i.e. the curve together with interior in \mathbb{C} .

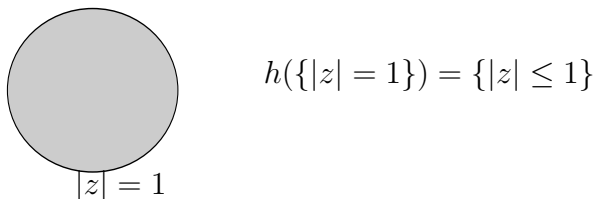


Figure 7: The holomorphic hull of the curve $\{|z| = 1\}$ in \mathbb{C}

Proof: By Maximum Modulus Principle, the points z such that $\{|z| \leq 1\}$ belong to the holomorphic hull.

Take a point z_0 so that $|z_0| > 1$. By considering the global holomorphic function z we see that this point does not belong to the holomorphic hull. \square

Example 5.2. Consider a curve γ

$$\{(z, w) \in \mathbb{C}^2 \mid |z| = 1, z = \bar{w}\}.$$

Then $h_{\mathbb{C}^2}(\gamma) = \gamma$.

Proof: The function $f(z, w) = zw - 1$ is equal to zero on γ . Therefore, $h(\gamma) \subset \{f = 0\}$. Take a point $(z_0, w_0) \in \mathbb{C}^2$.

- If $|z_0| > 1$. Then

$$|z_0| > \max\{|z| : (z, w) \in \gamma\}$$

Function z is a global holomorphic function. Therefore, the point (z_0, w_0) does not belong to $h(\gamma)$.

- If $|z_0| < 1$, then $|w_0| > 1$.

$$|w_0| > \max\{|w| : (z, w) \in \gamma\}$$

Therefore, the point (z_0, w_0) does not belong to $h(\gamma)$.

- If $|z_0| = 1$, then $z_0 = \bar{w}_0$. So $(z_0, w_0) \in \gamma$.

Thus, $h(\gamma) = \gamma$ □

Definition 5.2. Complex analytic manifold X of dimension n is said to be a *Stein manifold* if

1. for every compact set K its holomorphic hull $h(K)$ is also compact;
2. If z_1 and z_2 are two different points of X , then $f(z_1) \neq f(z_2)$ for some $f \in \mathcal{O}(X)$;
3. For every $z \in X$, one can find n functions $f_1, \dots, f_n \in \mathcal{O}(X)$ which form a coordinate system at z .

Fact 5.1. \mathbb{C}^n is a Stein manifold.

Fact 5.2. Every closed submanifold of a Stein manifold is a Stein manifold.

In fact there is the Embedding Theorem for Stein manifolds.

Fact 5.3. Every Stein manifold can be holomorphically embedded as a closed submanifold into \mathbb{C}^N .

Below we give one more equivalent definition of a Stein manifold in terms of a plurisubharmonic function, that is often used in practice.

Definition 5.3. A function φ defined in an open set $\Omega \subset \mathbb{C}^n$ with values in $[-\infty, +\infty)$ is *plurisubharmonic* if

1. it is semicontinuous from above.
2. For an arbitrary z and $w \in \mathbb{C}^n$, the function $\tau \rightarrow \varphi(z + \tau w)$ is subharmonic in the part of \mathbb{C} where it is defined.

Fact 5.4. A function $\varphi \in C^2(\Omega)$ is plurisubharmonic if and only if

$$\sum_{j,k=1}^n \partial^2 \varphi(z) / \partial z_j \partial \bar{z}_k w_j \bar{w}_k \geq 0, \quad (2)$$

where $z \in \Omega$, $w \in \mathbb{C}^n$.

Definition 5.4. Function φ is strictly plurisubharmonic if the inequality (2) is strict.

The notion of plurisubharmonicity does not depend on the choice of holomorphic coordinates. Therefore, it is well defined on all complex manifolds.

Fact 5.5. A complex manifold X is a Stein manifold if and only if there exists a strictly plurisubharmonic function $\varphi \in C^\infty(X)$ such that

$$\Omega_c = \{z \mid z \in X, \varphi(z) < c\} \Subset X$$

for any real number c . The sets $\hat{\Omega}_c$ are the $\mathcal{O}(X)$ -convex.

5.2 Complex foliations

Definitions 5.5-5.9 are from [15]. They are scattered through out the text, so we provide them here for the convenience of the reader. Definition 5.10, 5.11 can be found in [21],[3] correspondingly.

Definition 5.5. Let \mathcal{F} be a foliation of a complex manifold X . Let $\gamma : [0, 1] \rightarrow X$ be a path on X . Let T_0 and T_1 be two transversal sections to \mathcal{F} , passing through $\gamma(0)$ and $\gamma(1)$ respectively. Then for any initial point $x \in T_0$, close to $\gamma(0)$, leaf-wise curves, starting from x , and staying close to γ , and arriving to T_1 , arrive at a well defined point $\Delta_\gamma(x)$. Thus, we obtain a map $\Delta_\gamma(x)$, which we call the *holonomy map*. If $\gamma : [0, s] \rightarrow X$ is a closed curve, and T is a transversal section to \mathcal{F} , passing through $\gamma(0)$. The map $\Delta_\gamma : T \rightarrow T$ is called the *holonomy map* as well.

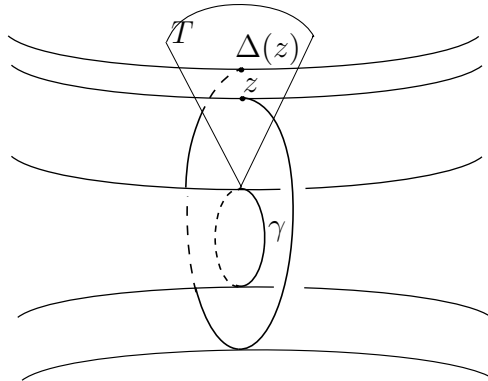


Figure 8: A holonomy map

Definition 5.6. A *complex cycle* is a nontrivial free homotopy class of loops on a leaf of a foliation. It is called *isolated* if it corresponds to an isolated fixed point of its holonomy map. It is *hyperbolic* if its holonomy map is hyperbolic, i.e. its linearization is non-degenerate, and the eigenvalues of the linearization do not belong to the unit circle.

Definition 5.7. Let γ be a hyperbolic cycle. The holonomy map Δ_γ has stable and unstable manifolds m_1^{loc}, m_2^{loc} . The union of leaves that pass through m_1^{loc} and m_2^{loc} are called stable and unstable manifolds of γ .

Definition 5.8. A singular point is called *complex hyperbolic* if it is non-degenerate and the ratio of any two eigenvalues is not real.

In this paper we work only with complex hyperbolic singular points. So we reserve the word "hyperbolic" to complex hyperbolicity.

Note that complex hyperbolicity plays a similar role for the theory of complex vector fields as hyperbolicity for the theory of real vector fields. In particular, if the point is complex hyperbolic, then the phase portrait of the vector field in the neighborhood of a singularity is homeomorphic to the phase portrait of its linearization [3]. See ([15], Section 29) for thorough consideration of properties of complex hyperbolic points.

Definition 5.9. A *complex separatrix* of a singular holomorphic foliation \mathcal{F} at a singular point $a \in \Sigma(\mathcal{F})$ is a local leaf $L \subset (U, a) \setminus \Sigma$, whose closure $L \cup a$ is a germ of an analytic curve.

Definition 5.10. A *saddle connection* is a common separatrix of two singular points. See Fig.2

Definition 5.11. Suppose a is a hyperbolic singular point of the foliation \mathcal{F} . Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of a . Let l be a line passing through the origin in \mathbb{C} . Let $\lambda = (\lambda_{i_1}, \dots, \lambda_{i_k})$ be the eigenvalues of a that lie on one side of the line l . Let α_λ be a subspace spanned by the eigenspaces of all elements of λ . The local *strongly invariant manifold* M_λ^{loc} is a manifold tangent to α_λ . The global strongly invariant manifold M_λ is obtained by taking the union of leaves that belong to the local strongly invariant manifold.

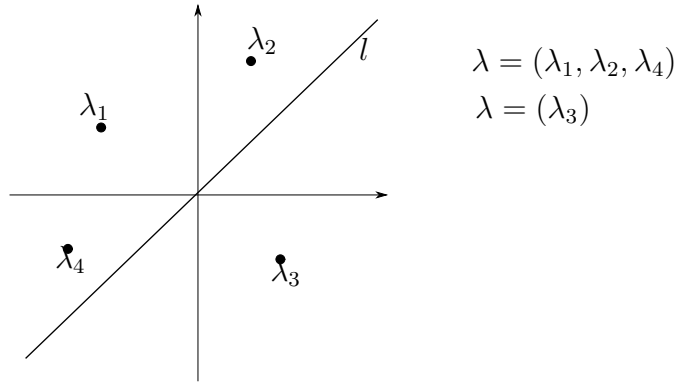


Figure 9: A line, separating eigenvalues.

Strongly invariant manifolds exist [15]. The proof can be easily modified to show that they depend holomorphically on a vector field (on a foliation).

Suppose that v is a vector field that determines a foliation locally. Strongly invariant manifolds are stable and unstable manifolds of the time-one map Φ_{cv}^1 of the vector field cv , where $c \in \mathbb{C}^*$ is taken so that l becomes the imaginary axis. If one considers the real flow of the vector field cv , then locally strongly invariant manifolds coincide with stable and unstable manifolds [3].

5.3 Holomorphic vector bundle associated to a foliation

Take a 1-dimensional singular holomorphic foliation \mathcal{F} of a Stein manifold M . One can naturally associate a linear bundle $B_{\mathcal{F}}$ to \mathcal{F} .

Notice that a 1-dimensional holomorphic foliation with singular locus of codimension 2 is locally determined by a holomorphic vector field [15]. Consider a covering of a Stein manifold by open contractible sets U_i . On each set U_i the foliation is determined by a holomorphic vector field v_i . For a pair of intersecting sets U_i and U_j define a function $g_{ij} = v_i/v_j$. This function is well-defined on $(U_i \cap U_j) \setminus \{v_j = 0\}$. The set $\{v_j = 0\}$ has codimension 2. Therefore, g_{ij} can be extended to $U_i \cap U_j$.

The same way $g_{ji} = v_j/v_i$ can be extended to a well-defined function on $U_i \cap U_j$.

$$g_{ij}g_{ji} = 1 \quad \Rightarrow \quad g_{ij}|_{U_i \cap U_j} \neq 0$$

The set of functions $\{g_{ij}\}$ form a 2-cocycle, therefore, they define a linear bundle.

Lemma 5.1. *1-dimensional singular holomorphic foliation \mathcal{F} of a Stein manifold X is determined by a global section of the vector bundle $TX \otimes B_{\mathcal{F}}$.*

Proof: Lemma follows from the construction of $B_{\mathcal{F}}$. □

If $H_2(X, \mathbb{Z}) = 0$, then each foliation on X is determined by a global vector field. In particular, this holds for foliations on \mathbb{C}^n .

5.4 Topology of uniform convergence on compact non-singular sets

The description of topology on the space of foliations in \mathbb{C}^n is given for example in [11]. Let X be a Stein manifold. We fix its compact exhaustion:

$$K_1 \Subset \cdots \Subset K_n \cdots \Subset X,$$

where K_1, \dots, K_n are compact subsets of X , closures of open connected subsets of X ;

$$\cup_n K_n = X.$$

Let d_1 be a metric on X and d_2 be a metric on the projectivization of its tangent bundle PTX . A basis of neighborhoods of the foliation \mathcal{F} is formed by

$$U_{n,\varepsilon,\delta} = \left\{ \mathcal{G} \mid \begin{array}{l} \mathcal{G} \text{ is nonsingular in } K_{\varepsilon,n} = K_n \setminus U_{\varepsilon}(\Sigma(\mathcal{F})) \text{ and the tangent directions} \\ \text{to the foliations } \mathcal{F} \text{ and } \mathcal{G} \text{ are } \varepsilon\text{-close on } K_{\varepsilon,n} \end{array} \right\}.$$

Note that the obtained topology does not depend on the choice of compact exhaustion and the choice of metrics d_1 and d_2 . The set of foliations of X has countably many connected components, parametrized by Chern classes of the linear bundles, associated to the foliations.

The set of sections of $TX \otimes B_{\mathcal{F}}$ is equipped with the topology of uniform convergence on compact sets. (See for the description of the linear bundle $B_{\mathcal{F}}$.) The map from the space of sections to the space of foliations is continuous.

5.5 Multiplicity

We consider analytic subsets A of a polydisk \bar{D}^n , i.e. we assume that A is an analytic subset of some neighborhood of D^n . Suppose that A is given by a system of n equations

$$f_1 = \cdots = f_n = 0$$

Assume that A is k -dimensional. We want to define the multiplicity of A which does not increase under perturbations.

Lemma 5.2. *There are only finitely many strata of maximal dimension.*

Proof: The number of strata is locally finite [5]. Since A is an analytic subset of \bar{D}^n , it is globally finite. \square

Let A_1, \dots, A_m be the strata of maximal dimension.

Take a smooth point $z \in A_i$. Consider a transversal section T to A_i at the point z . Let $\tilde{f}_1, \dots, \tilde{f}_n$ be the restriction of f_1, \dots, f_n to T . The point z is an isolated solution of the system

$$\tilde{f}_1 = \cdots = \tilde{f}_n = 0.$$

Definition 5.12. Let z be an isolated point of a system of equations

$$\tilde{f}_1 = \cdots = \tilde{f}_n = 0,$$

defined in $n-k$ -dimensional polydisk D^{n-k} . The multiplicity $m(z)$ of a point z is the dimension of

$$\mathcal{O}_{D^{n-k},z} / \langle \tilde{f}_1, \dots, \tilde{f}_n \rangle,$$

where $\mathcal{O}_{D^{n-k},z}$ is the local ring of $z \in D^{n-k}$, i.e. functions, regular in a neighborhood of $z \in D^{n-k}$; $\langle \tilde{f}_1, \dots, \tilde{f}_n \rangle$ is the ideal in $\mathcal{O}_{D^{n-k},z}$ generated by $\tilde{f}_1, \dots, \tilde{f}_n$.

Lemma 5.3. *The multiplicity does not increase under perturbations, i.e. if z'_1, \dots, z'_m are isolated solutions of a perturbed system in a neighborhood of a point z , then*

$$\sum_{i=1}^m m(z'_i) \leq m(z).$$

Proof: In ([1], Chapter 2, 5.7) it is proved for $k = 0$. In general case the proof goes the same way. \square

Definition 5.13. The multiplicity of $z \in A_i$ is the multiplicity the point z as an isolated solution of $\tilde{f}_1 = \cdots = \tilde{f}_n = 0$.

It is easy to see that multiplicity does not depend on the choice of a generic point and a transversal section T .

Definition 5.14. The multiplicity of a stratum A_i is the multiplicity of a generic point. The multiplicity of A is the sum of multiplicities of A_i .

Lemma 5.4. *The multiplicity of A does not increase under perturbations, i.e. let $A'_1, \dots, A'_{m'}$ be strata of a perturbed system, then*

$$\sum_{i=1}^{m'} m(A'_i) \leq m(A).$$

Proof: Let T_1, \dots, T_m be transversal sections to A_i 's at generic points. Every A'_i intersect at least one of the sections T_1, \dots, T_m . One can also assume that T_i 's meet A_i 's transversally. On each transversal section the result follows from the Lemma 5.3.

□

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